Simultaneous Analysis and Design

Raphael T. Haftka*

Virginia Polytechnic Institute and State University, Blacksburg, Virginia

Optimization techniques are increasingly being used for performing nonlinear structural analysis. The development of element-by-element preconditioned conjugate gradient techniques is expected to extend this trend to linear analysis. Under these circumstances the structural design problem can be viewed as a nested optimization problem. The present paper suggests that there are computational benefits to treating this nested problem as a large single optimization problem. That is, the response variables (such as displacements) and the structural parameters are all treated as design variables in a unified formulation which performs simultaneously the design and analysis. Two examples are used for demonstration. A 72-bar truss is optimized subject to linear stress constraints and a wing-box structure is optimized subject to nonlinear collapse constraints. Both examples show substantial computational savings with the unified approach as compared to the traditional nested approach.

Introduction

S TRUCTURAL optimization was initially based on the calculus of variations. A typical problem was solved by obtaining the Euler-Lagrange optimality differential equations and solving them simultaneously with the differential equations of the structural response. This approach is still used today for the optimization of individual structural elements such as beam-columns;[1] however, for built-up structures modeled by finite elements, a nested approach is typical. Resizing rules based on optimality criteria require that the structural response be calculated repeatedly for each set of trial structural design variables (see, for example, Ref. 2). This preference for the nested rather than the simultaneous approach is probably due to the simplicity of the structural resizing rules which are possible when the structural response is known. This simplicity contrasts with the difficulty of solving the large systems of nonlinear algebraic equations which are obtained from a simultaneous formulation.

In the last 25 years direct search methods have been gaining ground as the standard for structural optimization. These techniques are commonly used in a nested approach. That is, the structural analysis equations are repeatedly solved during each design iteration. Part of the reason for the popularity of the nested approach is that the structural analysis equations are solved by techniques which are quite different from those used for the design optimization. An exception is the design of a structure subject to constraints on its collapse load. There, the analysis problem ("limit analysis") is often approximated as a linear program and solved by the simplex method. The structural design problem in that case ("limit design") is easily formulated as a single linear program with the element forces and structural parameters both treated as design variables (e.g., Ref. 3).

In the late 1960s, Fox, Schmit and their co-workers[4-8] tried to integrate structural analysis and design by employing conjugate gradient (CG) minimization techniques for solving linear structural analysis problems. They found that the optimization methods were not competitive with the traditional direct Gaussian elimination techniques. More recently techniques for unconstrained minimization have become more efficient and their application to structural analysis has

become more feasible (e.g., Ref. 9). The emergence of the preconditioned conjugate gradient techniques (e.g., Ref. 10) and the element-by-element (EBE) formulations of Hughes and co-workers[11] make CG methods particularly attractive for structural analysis.

In view of the increasing use of optimization methods for structural analysis, there is merit in considering again the simultaneous approach to analysis and design. When optimization techniques are used for structural analysis the design problem becomes a nested optimization problem. Such nested problems can, indeed, often benefit computationally from a single level treatment that disregards the nested structure.[12] The present paper investigates the use of the simultaneous analysis and design approach for linear and nonlinear problems. An EBE preconditioned CG method is applied to a linear analysis problem, and Newton's method is used for design subject to a nonlinear collapse load constraint. Substantial computational savings are demonstrated for both examples.

Linear Structural Analysis and the Conjugate Gradient Method

In the linear case the finite element discretization of a structure typically leads to a system of linear equations

$$R \equiv KU - F = 0 \tag{1}$$

where K is a stiffness matrix, U a displacement vector, and F a load vector. Equation (1) is usually solved by Gaussian elimination, but it can also be solved by minimizing the function

$$f(\tilde{U}) = \tfrac{1}{2}(\tilde{U} - U)^T K(\tilde{U} - U) \tag{2}$$

where U is the solution of Eq. (1). Clearly, the minimum of f is obtained for $\tilde{U} = U$. The function $f(\tilde{U})$ may be minimized even though U is unknown because its gradient G may be calculated without using U

$$G = \nabla f = K(\tilde{U} - U) = K\tilde{U} - F \tag{3}$$

Applying the CG method to the minimization of f is often very expensive because the second derivative matrix of f, which is K, usually has a high condition number. This problem can be remedied if we have an approximation B to K. Then, B can be used to "precondition" the problem. The

preconditioned CG algorithm is summarized in the Appendix.

While any reasonable approximation to the inverse of K would solve the ill-conditioning problem, the method is economical only if B^{-1} is inexpensive to calculate compared to K^{-1}. When K is very poorly banded, one approach is to obtain B by an incomplete factorization of K (e.g., Ref. 13). Another attractive approach is the element-by-element (EBE) technique of Hughes.[11] Reference 11 presents several formulations for obtaining B, and the one used herein is given in the Appendix.

Simultaneous Analysis and Design: Linear Case

The structural design problem may be formulated as:
Find X to minimize $m(X)$ subject to

$$g_j(U,X) \geq 0 \qquad j=1,...,m \qquad (4)$$

where m is an objective function, X a vector of design parameters, and g_j constraint functions such as stress and displacement constraints. In addition, the displacement vector U must satisfy Eq. (1), where both K and F may depend on X. The optimization problem is usually solved by repeatedly calculating U and its derivatives with respect to the components of X, x_j. U is calculated from Eq. (1) and dU/dx_j is calculated either by differentiating Eq. (1) or by finite differences. Based on U and its derivatives, the constraint functions g_j and their derivatives can be evaluated and a numerical optimization technique can use this information to improve X.

This approach is wasteful because the displacement vector U is calculated exactly for each trial design X. A simultaneous approach treats X and U equally as deisgn variables and solves the following expanded problem:
Find X,U to minimize $m(X)$ subject to

$$g_j(X,U) \geq 0 \qquad j=1,...,m$$

and

$$R = KU - F = 0 \qquad (5)$$

The equations of equilibrium are treated here as equality constraints.

The optimization technique used here to solve the problem of Eq. (5) is a penalty function technique. An application of a standard penalty technique replaces Eq. (5) by

$$\text{minimize } \phi(X,U,r) = cm(X) + r \sum_{j=1}^{m} p[g_j(X,U)] + \frac{c_l}{\sqrt{r}} R^T R$$

for $r = r_1, r_2,...,$ *where*

$$r_i \rightarrow 0 \qquad (6)$$

where

$$p(g) = 1/g_0 [(g/g_0)^2 - 3g/g_0 + 3] \qquad (7)$$

is an extended interior penalty function[14] with g_0 being a transition parameter. The constants c and c_l are chosen to balance the contribution of the objective function, inequality constraints and equality constraints to ϕ (see Ref. 15 for details). If the minimization of ϕ is accomplished by the CG method, then the $R^T R$ term presents a problem

$$R^T R = (KU-F)^T (KU-F) \qquad (8)$$

so that the second-derivative matrix of $R^T R$ with respect to U is $K^T K$. Unfortunately, the condition number of $K^T K$ is the square of that of K so that the optimization of ϕ by the

CG method would proceed extremely slowly. It is tempting to replace the term $R^T R$ by the function f defined by Eq. (2). However, while the derivatives of f with respect to the displacement can be calculated [see Eq. (3)] without knowing the exact displacement vector U, the derivatives of f with respect to X cannot.

It is due to the ill-conditioning of ϕ that the idea of simultaneous analysis and design has not been pursued in the past. The solution proposed here is to replace the term $R^T R$ by $R^T B^{-1} R$ so that

$$\phi(X,U,r) = cm(X) + r \sum_{j=1}^{m} p[g_j(X,U)] + \frac{c_l}{\sqrt{r}} R^T B^{-1} R \qquad (9)$$

Because B is an approximation to K^{-1} the $R^T B^{-1} R$ term has conditioning similar to f.

Another way of avoiding the ill-conditioning due to the $R^T R$ term is to use Newton's method instead of the CG method. This option is explored in the next section.

Simultaneous Analysis and Design: Nonlinear Case

The formulation described in the previous section can also be used in the case where R is a nonlinear function of U. In that case, B would be an approximation to the Jacobian of R. To broaden the scope of the discussion, a different nonlinear formulation based on collapse techniques[15] is presented next. This formulation contains a combination of linear and nonlinear analyses and demonstrates the option of solving the nonlinear analysis problem simultaneously with the design problem while treating the linear analysis sequentially.

Reference 15 considers, a structure which is subject to some load conditions when it is intact, and to other load conditions when it is damaged. For the sake of simplicity only one load case in each condition is treated here. In its undamaged condition the structure is subject to the following.
Displacement constraints

$$g_{di}(X,U) \geq 0 \qquad i=1,...,n_d \qquad (10)$$

Stress constraints

$$g_{si}(X,U) \geq 0 \qquad i=1,...,n_s \qquad (11)$$

Buckling constraints

$$g_{bi}(X,U) \geq 0 \qquad i=1,...,n_b \qquad (12)$$

The displacement U can be calculated from the linear analysis

$$KU = F_I \qquad (13)$$

where F_I is the load applied to the undamaged structure. For the damaged structure, large deformation and post-buckling response can be tolerated as long as the structure does not collapse. In this case, an approximate calculation of the collapse load can be performed by assuming that the equations of compatibility can be disregarded. That is, we assume that after yielding or buckling the internal loads begin to redistribute, with yielded and buckled elements becoming "soft" and undergoing large deformations. The situation is idealized here by assuming zero post-buckling or post-yielding stiffness. That is, the yielded or buckled element continues to carry the load that it carried at the onset of buckling or yielding, and that load does not increase with additional deformation. The collapse load is reached when no amount of internal load redistribution can balance the applied loads. To take advantage of the above assumption for calculating the collapse load, the element forces are used as the unknowns in the equations of equilibrium instead of the

displacements. That is, the equations of equilibrium are written as

$$ET = fF_D \tag{14}$$

where E is a matrix which depends only on the initial geometry of the structure, T a vector of element internal loads, F_D the design load for the damaged structure, and f a safety factor. For statically indeterminate structures the matrix E is rectangular and it is not possible to determine T uniquely from Eq. (14). Instead, based on the assumptions that we made, the structure will not collapse if it is possible to find a solution to Eq. (14) for $f = 1$ such that the stress and buckling constraints are satisfied. These constraints are rewritten in terms of T as

$$g_{si}(T) \geq 0 \qquad i = 1,...,n_s \tag{15}$$

$$g_{bi}(T) \geq 0 \qquad i = 1,...,n_b \tag{16}$$

The constraint that the structure does not collapse at the applied load may be written as

$$f_{max} \geq 1 \tag{17}$$

where f_{max} is the maximum value of f that may be obtained by selecting T that satisfies Eqs. (14-16).

In Ref. 15 the design problem was posed as:
Find X to minimize $m(X)$ subject to

$$g_{di}(U) \geq 0 \qquad i = 1,...,n_d$$

$$g_{si}(U) \geq 0 \qquad i = 1,...,n_s$$

$$g_{bi}(U) \geq 0 \qquad i = 1,...,n_b \tag{18}$$

and

$$f_{max} \geq 1$$

where U was obtained by a direct solution (Gaussian elimination) of Eq. (13) and f_{max} was obtained by solving a maximization problem. The calculation of the displacement vector U and the collapse margin of safety f_{max} represents the structural analysis and had to be repeated many times as the optimization with respect to X proceeded. However, the calculation of f_{max} was by far more costly than the calculation of U. For this reason, the technique of simultaneous analysis and design is applied here only to the collapse constraint. That is, the design problem is rewritten as:
Find X,T to minimize $m(X)$ subject to

$$g_{di}(U) \geq 0 \qquad i = 1,...,n_d$$

$$g_{si}(U) \geq 0 \qquad i = 1,...,n_s$$

$$g_{bi}(U) \geq 0 \qquad i = 1,...,n_t$$

$$ET = fF_D$$

$$g_{si}(T) \geq 0 \qquad i = 1,...,n_s$$

$$g_{bi}(T) \geq 0 \qquad i = 1,...,n_b$$

$$f \geq 1 \tag{19}$$

where U is calculated from Eq. (13). The optimization problem Eq. (19) is solved by a penalty function approach by

minimizing an augmented objective function

$$\phi(X,T,r) = cm(X) + r\sum_{i=1}^{n_d} p[g_{di}(U)]$$

$$+ r\sum_{i=1}^{n_s} \{p[g_{si}(U)] + p[g_{si}(T)]\}$$

$$+ r\sum_{i=1}^{n_b} \{p[g_{bi}(U)] + p[g_{bi}(T)]\}$$

$$+ \frac{c_1}{\sqrt{r}}(ET - fF_D)^T(ET - fF_D) + rp(f - 1) \tag{20}$$

To overcome the ill-conditioning coming from the penalty due to the equations of equilibrium, Newton's method rather than a preconditioned CG method is employed. This is motivated by the fact that unlike the matrix K, the matrix E is not a function of the design variables so that second derivatives are easier to calculate. The second derivatives of the penalty terms for the undamaged structure are approximated by using only first derivatives of the constraints (see Ref. 14).

Results and Discussion

Linear Analysis

The example used for demonstrating the usefulness of an EBE preconditioned CG method for the linear case is a 72-bar truss shown in Fig. 1. The loading and the stress allowables are given in Table 1. The regular CG method and the EBE preconditioned CG method are compared in Fig. 2, which shows the convergence of the stress in member 1. It is clear from Fig. 2 that the convergence of the preconditioned CG method is much faster than that of the regular CG method.

Next, the truss is optimized subject to stress constraints. The cross-sectional areas of the 72 members are the design variables. The optimization is performed three different ways. First, the traditional sequential approach is employed using a standard truss finite element analysis to calculate stresses and the NEWSUMT optimization program[16] to optimize the cross-sectional areas. The derivatives of the con-
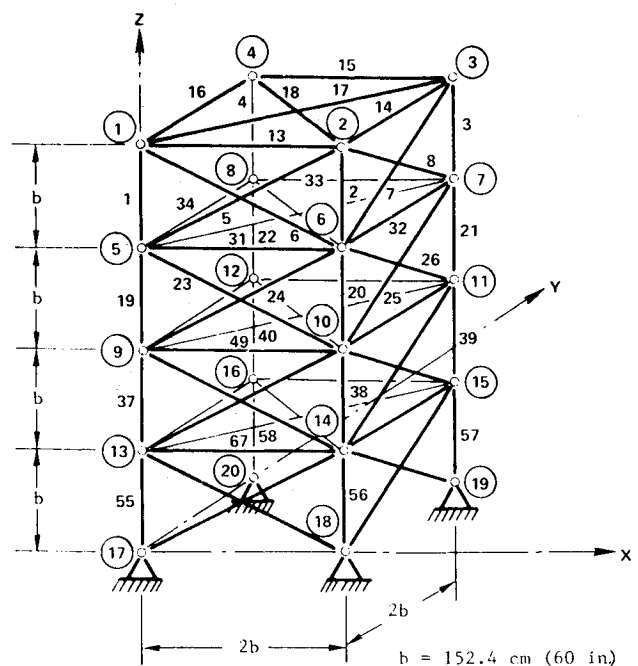

Fig. 1   72-bar space truss.

straints are calculated by finite differences. The second optimization is carried out by a CG package based on Beale's restarted CG algorithm[17] applied to the penalty function formulation of Eq. (6). The design variables are both the 72 cross-sectional areas and the 48 nonzero displacement components. Finally, the same optimization is repeated with the modified formulation of Eq. (9) where the EBE approximate matrix B was calculated for the initial design and not updated as the design changed. The results of the three optimizations are summarized in Table 2. As can be seen from Table 2, the use of the EBE-generated approximate inverse reduces the number of constraint evaluations and CPU time by an order of magnitude. The comparison with the traditional sequential optimization approach is more difficult because its cost is inflated by the use of finite difference derivatives, and the optimization method is different. Even so, it does appear that the simultaneous optimization approach could be expected to be better than the traditional sequential approach.

The results of the 72-bar truss were obtained for a single-load case. When multiple-load cases need to be considered, the CG method is at a disadvantage compared to direct methods. In such cases the simultaneous approach could rely on Newton's method. The use of Newton's method is demonstrated now for a nonlinear problem.

Nonlinear Analysis

Reference 15 discusses the design for damage tolerance of the wing-box structure shown in Fig. 3. The box is 3.56 m

(140 in.) long, 2.24 m (88 in.) wide, and 38 cm (15 in.) deep. As shown in Fig. 3, the wing box is clamped at the root and a variable load is applied at the tip. The loads applied at the four tip nodes are 8,163, 17,000, 17,000, and 34,000 kg (18,000, 37,500, and 75,000 lb). The upper and lower wing-skin panels are modeled by membrane elements, and the webs of the ribs and spars are modeled by shear web elements. The spar caps and vertical posts at the rib-spar intersections are modeled by rod elements. The wing is assumed to be made of 7075 aluminum alloy.

The finite element model employs a single quadrilateral membrane element to represent a skin panel bounded by ribs and spars. The model has 32 grid points and 75 finite elements. Design variables are distributed so that there is one design variable for the thickness of each cover skin between
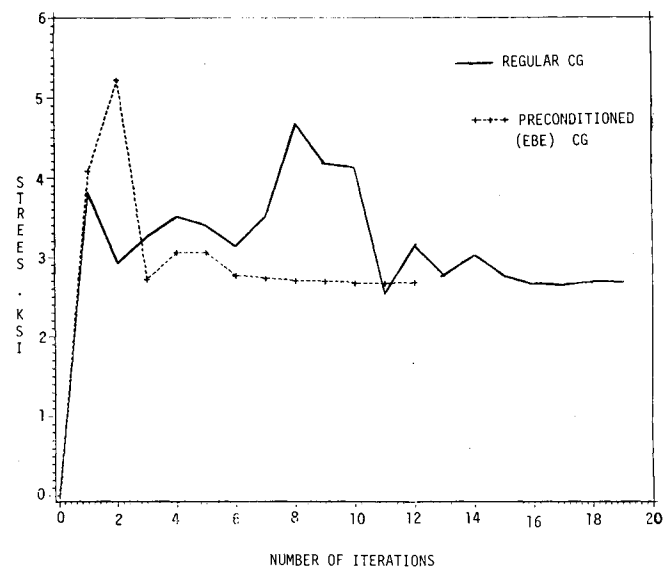

Fig. 2　Convergence of stress in member 1 of 72-bar truss.

Table 1　Definition of 72-bar space truss

Material	Aluminum
Young's modulus	$E = 6.9 \times 10^7$ kN/m^2 (10^7 psi)
Specific mass	$\rho = 2768$ kg/m^3 (0.1 lbm/in.^{3})
Allowable stress	$\sigma_a = \pm 172,375$ kN/m^2 ($\pm 25,000$ psi)
Minimum area	$D^{(L)} = 0.645$ cm^2 (0.1 in.^{2})
Uniform initial area	$D^{(0)} = 6.45$ cm^2 (1.0 in.^{2})

Nodal loading

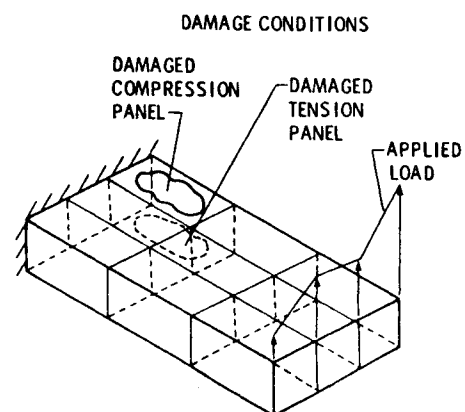| Node	Load components N, lbf		
	X	Y	Z
1	22,240 (5,000)	22,240 (5,000)	−22,240 (−5,000)
2	0	0	−22,240 (−5,000)
3	0	0	−22,240 (−5,000)
4	0	0	−22,240 −5,000

DAMAGE CONDITIONS

Fig. 3　Wing-box structure.

Table 2　Comparison of sequential and simultaneous analysis-design optimization for 72-bar truss

Approach	Solution method	No. of r values	No. of constraint evaluations	CPU time IBM 3081, s	Final weight, lb
Sequential	Newton	14	4733 (4464 for derivatives)	185.9	95.6
Simultaneous	CG	12	13347	150.0	96.7
Simultaneous	EBE-CG	12	665	8.8	96.7

adjacent ribs (i.e., constant chordwise distribution), one design variable assigned to the thickness of each spar web between ribs, one design variable assigned to the area of each spar cap between ribs, and one design variable assigned to the thickness of each rib. The cross-sectional areas of the vertical posts at the rib-spar intersections are held constant. The total number of design variables is 45. The design constraints are stress constraints [503 MPa (73,000 psi) allowable stress, using the von Mises yield criterion], buckling constraints for panels and shear webs, and side constraints. The side constraints are 0.5 mm (0.02 in.) minimum gage for skin panels, 0.65 cm^2 (0.1 in.^{2}) minimum area for spar caps, and 3.9 cm^2 (0.6 in.^{2}) maximum area for spar caps.

The computation time with the traditional sequential approach was 7000 CPU/s on a CDC Cyber-173 computer. The simultaneous approach required only 1600 CPU/s. The final designs had similar design variable distribution with less than 1% difference in mass (see Ref. 15 for details of final design).

Concluding Remarks

Nonlinear structural analysis is usually performed by iterative methods. Recently iterative algorithms are becoming competitive even for linear analysis. The present paper investigates the possibility of interfacing the analysis iterations with design optimization iterations and combining the two in a single optimization problem.

For linear problems a 72-bar truss example is used to show that the element-by-element approximate inverse of the stiffness matrix can be used to speed the convergence of the simultaneous analysis-design convergence by an order of magnitude.

For a wing-box damage-tolerance example employing collapse techniques, the simultaneous approach reduces the computation time by better than a factor of four.

The two examples demonstrate the potential of the simultaneous analysis and design approach. More work is needed to generalize the technique for nonlinear analysis applications.

Appendix: The EBE Preconditioned CG Method

The flow chart for solving the linear system $KU = F$ is

Step 1: Initialization $m = 0$, $U_0 = 0$, $R_0 = F$,

$$P_0 = Z_0 = B^{-1}R_0$$

Step 2: $\alpha_m = R_m^T Z_m / P_m^T K P_m$

Step 3: $U_{m+1} = U_m + \alpha_m P_m$

Step 4: $R_{m+1} = R_m - \alpha_m K P_m$

Step 5: Convergence check

$$||R_{m+1}|| < \delta?$$

Yes: Return

No: Continue

Step 6: $Z_{m+1} = B^{-1} R_{m+1}$

Step 7: $\beta_m = R_{m+1}^T Z_{m+1} / R_m^T Z_m$

Step 8: $P_{m+1} = Z_{m+1} + \beta_m P_m$

Increment m and go to step 2

The matrix B is an approximation of K obtained[11] as

$$B = W^{1/2} C W^{1/2}$$

where W is a diagonal scaling matrix taken here to be the diagonal of the matrix K. The matrix C is a product of factored element matrices given subsequently. For each finite element we generate the factorization

$$I + \bar{K}^e = L_p^e D_p^e (L_p^e)^T$$

where

$$\bar{K}^e = W^{-1/2}(K^e - D^e)W^{-1/2}$$

K^e is the element stiffness matrix and D^e a diagonal matrix composed of the diagonal terms of K^e. The matrix C is then given as

$$C = \left[\prod_{e=1}^{n_{el}} L_p^e \right] \left[\prod_{e=1}^{n_{el}} D_p^e \right] \left[\prod_{e=n_{el}}^{1} (L_p^e)^T \right]$$

and n_{el} is the number of finite elements. The factorization of C is performed completely at the element level.

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